HARNACK TYPE INEQUALITY FOR POSITIVE SOLUTION OF SOME INTEGRAL EQUATION

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ABSTRACT. In this paper, we establish some Harnack type inequalities satisfied by positive solutions of nonlocal inhomogeneous equations arising in the description of various phenomena ranging from population dynamics to micro-magnetism. For regular domains, we also derive an inequality up to the boundary. The main difficulty in such context lies in a precise control of the solutions outside a compact set and the existence of local uniform estimates. We overcome this problem by proving a contraction result which makes the L^1 norms of the solutions on two compact sets $\omega_1 \subset\subset \omega_2$ equivalent. We also construct the principal positive eigenfunctions associated to particular nonlocal operators by using the corresponding Harnack type inequalities.

1. INTRODUCTION AND MAIN RESULTS

In this paper, we investigate the existence of a Harnack type inequality for positive continuous solutions of

(1.1)
$$\mathcal{L}[u] = 0$$

where the operator is defined by

(1.2)
$$\mathcal{L}[u] := \int_{\Omega} k(x,y)u(y)\,dy - b(x)u,$$

with $\Omega \subset \mathbb{R}^d$, $k \geq 0$ and $b(x) \in C(\Omega)$. Precise assumptions on Ω, k and b will be given later on.

Such type of linear operator has been widely used to describe the dispersal of a population in its environment in the following sense. As stated in [25, 26, 32], if $u(y,t)$ is thought of as a density at a location y at a time t and $k(x,y)$ as the probability distribution of jumping from a location y to a location x, then the rate at which the individuals from all other positions are arriving to the location x is

$$\int_{\Omega} k(x,y)u(y,t)\,dy.$$

On the other hand, the rate at which the individuals are leaving the location x to travel to all other places is $-b(x)u(x,t)$. The operator $\mathcal{L}$ is called nonlocal since the behaviour of a function at a given point depends on its values at points some distance away rather than just nearby ones.

In the past few years, there has been an intense interest to use such nonlocal operators to model problems in mathematical physics or in ecology, see, among other references, [22, 26, 32, 34, 38]. In particular, much attention has been devoted to the study of the nonlocal reaction diffusion equation

$$(1.3) \qquad \frac{\partial u}{\partial t} = \mathcal{L}[u] + f(x, u(x,t)) \quad \text{in} \quad \Omega \times \mathbb{R}^+,$$

where the usual elliptic diffusion operator is replaced by the operator $\mathcal{L}$, see for example [5, 16, 32, 34]. Problems related to (1.3) with a homogeneous nonlinearity have been widely treated in the literature when $b(x) \equiv 1$ and the kernel take the form of a convolution operator, i.e. $k(x,y) = J(x - y)$ with J a probability density. For instance, the works in the references [5, 18, 19, 21, 23] are devoted to the study of travelling front solutions existing for the problem (1.3) with a homogeneous bistable or monostable nonlinearity, while [4, 13] deal with the study of the steady state solutions of the problem (1.3) with a logistic type, bistable or power-like nonlinearity. The particular instance of the parabolic problem in $\mathbb{R}^n$ when $f = 0$ is considered in [12, 17]. In the heterogeneous case (b and k general), few results are known. This is due mostly to the lack of compactness properties of $\mathcal{L} + \lambda$ or of its inverse. We quote [20, 32] which deal with a convolution kernel in a periodic environment with a monostable type nonlinearity and [16] in the case of a linear heterogeneous dispersal process in $\mathbb{R}$.

When the kernel $k(x, y)$ is smooth and compactly supported, it is easy to see that, when applied to a smooth function, the operator $\mathcal{L}$ can be rewritten into the following form

$$\mathcal{L}[u] := \mathcal{M}[u] + \mathcal{R}[u]$$

with $\mathcal{M}$ an elliptic operator

$$(1.4) \qquad \mathcal{M}[u] := a_{ij}(x)\partial_{ij}u + b_i(x)\partial_i u + c(x)u,$$

and $\mathcal{R}$ an operator involving derivatives of higher order than in $\mathcal{M}$. Indeed, we have

$$\mathcal{L}[u] = \int_\Omega k(x,y)[u(y) - u(x)]\, dy - c(x)u,$$

with $c(x) := b(x) - \int_\Omega k(x,y)dy$. Hence, setting $z = x - y$ and performing a formal Taylor expansion of u in the integral, we obtain

$$\int_{x-\Omega} k(x, x - z)[u(x - z) - u(x)]\, dy = a_{ij}(x)\partial_{ij}u + b_i(x)\partial_i u + R[x, \partial_{ijk}u]$$

with $a_{ij}(x)$ and $b_i(x)$ defined by the following expressions

$$a_{ij}(x) = \frac{1}{2}\int_{x-\Omega} k(x, x - z)z_i z_j\, dz$$

$$b_i(x) = \int_{x-\Omega} k(x, x - z)z_i\, dz$$

and

$$R[x, \partial_{ijk}u] := \int_{x-\Omega} k(x, x - z)z_i z_j z_k \left(\int_0^1 \int_0^1 \int_0^1 st^2 \partial_{ijk}u(x - hstz)\, dh\, ds\, dt\right) dz.$$

Therefore, for suitable kernels k, it seems reasonable to expect that the operator $\mathcal{L}$ shares many of the properties of $\mathcal{M}$.

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For a uniform elliptic operator $\mathcal{M}$, it is well known that the positive solutions of the equation $\mathcal{M}[u] = 0$ are satisfying an Harnack inequality, see [24, 27]. That is to say

Theorem 1.1 (Harnack Inequality [24, 27]). *Let $\Omega \subset \mathbb{R}^n$ be a domain and let $\omega \subset\subset \Omega$ be a compact sub-domain. Then there exists a constant $C(\omega)$ such that for all positive smooth solutions u so that*

$$\mathcal{M}[u] = 0$$

we have

$$u(x) \leq Cu(y) \quad \text{for all} \quad x, y \in \omega.$$

Such estimate is an extremely important tool in the study of partial differential equations since they are providing in particular various key estimates in the analysis of the regularity of the solutions of PDE's. Moreover, such estimate plays a central role in the construction of a positive eigenfunction of $\mathcal{M}$, which is also an essential tool in the analysis of various nonlinear partial differential equations, see for example [7, 8, 9].

As observed in [16, 19, 32] for some particular choice of b and k, the principal eigenvalue λ_1 of $\mathcal{L}$ and its corresponding positive eigenfunction ϕ_1, that is

$$(1.5) \qquad\qquad \mathcal{L}[\phi_1] + \lambda_1 \phi_1 = 0 \quad \text{in} \quad \Omega,$$

are central in the analysis of the nonlocal reaction diffusion problems such as (1.3). However, due to the lack of compactness of $\mathcal{L} + \lambda$ or of its inverse, proving the existence of a principal eigenpair in this context is a very difficult task and no general result is known. As for elliptic operators, an Harnack type inequality for the positive solutions of such nonlocal equation (1.5) is expected to provide *a priori* estimates for the construction of eigenpairs of a general operator $\mathcal{L}$. It is therefore of great interest to investigate the existence of such estimates. For some particular kernel $k(x, y)$, the Harnack estimate for solutions of (1.1) are often obtained as a consequence of the harmonic property of the solution. Indeed, when $k(x, y)$ takes the following form

$$k(x, y) = \begin{cases} \frac{1}{|B(x, r(x))|} & \text{if } y \in B(x, r(x)) \\ 0 & \text{otherwise} \end{cases}$$

where $c_0 < r(x) < d(x, \complement\Omega)$ is a given function, then the positive solutions of (1.1) are harmonic functions. There exists a vast literature dealing with what is called in potential theory *"The Converse Mean Value Problem"*. The central question is then to find the conditions on $k(x, y)$ so that the solutions u of (1.1) are harmonic. For more details on this subject, we refer the interested reader to [6, 15, 28, 29, 31, 30, 36, 39] and references therein. For the above kernel, to our knowledge, Veech [39] was the first to obtain a Harnack type estimate for the positive solutions of (1.1) as a consequence of the restricted mean value property itself rather than the harmonic property of the solution. Later, Cornea and Veselý [14, 15] have extended the Harnack type estimate obtained by Veech to some more general kernels. More precisely, they have obtained the following

Theorem 1.2 ([14, 15]). *Let $k(x, y)$ be a kernel so that $\forall x \in \Omega$, there exists V_x and W_x, two compact neighbourhoods of x and two strictly positive constants m_x and M_x such that $W_x \subset \Omega$ and for any $y \in V_x$ we have*

$$(1.6) \qquad\qquad m_x \chi_{V_x} \leq \frac{1}{b(.)} k(., y) \leq M_x \chi_{W_x}$$

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where χ_A is the characteristic function of the set A. Then, for all compact set $\omega \subset \Omega$ there exists a constant $C(\omega, k)$ such that for all super-median function u (i.e. any function u satisfying $u(x) \geq \frac{1}{b(x)} \int_\Omega k(x,z)u(z)\,dz$ for all x) and any two points $x, y \in \omega$, one have

$$\frac{1}{b(x)} \int_\Omega k(x,z)u(z)\,dz \leq C\frac{1}{b(y)} \int_\Omega k(y,z)u(z)\,dz.$$

Some Harnack inequality for nonlocal operators have also been proved for singular kernel, essentially for the generators of pure jump processes, see for example [1, 2, 3, 10, 11, 33]. In this context, the operator $\mathcal{L}$ takes the form

$$(1.7) \qquad \mathcal{L}[u] := \int_\Omega k(x,y)[u(y) - u(x)]\,dy,$$

where $k(x,y) \sim \frac{1}{|x-y|^{d+\alpha}}$ with $\alpha > 0$. This particular structure implies some extra regularity for the solutions of (1.1) and plays an essential role in the derivation of the Harnack inequality.

In this work, we investigate the existence of a Harnack type inequality for a class of nonlocal operators $\mathcal{L}$ for which the condition on the kernel (1.6) is not always verified and when the compact set ω can touch the boundary $\partial\Omega$ when it exists. More precisely, we study the existence of a Harnack type inequality for the positive solutions of the operator defined in (1.2), for any positive function $b(x)$ and when the kernel $k(x,y)$ takes the form introduced by Cortazar *et al.* in [16]:

$$(1.8) \qquad k(x,y) = J\left(\frac{x-y}{g(y)}\right)\frac{1}{g^n(y)},$$

where J is a probability density and the function g is bounded and non negative. We are particularly interested in finding some simple conditions on J, g, b and Ω such that a Harnack type inequality holds for the positive solutions of the following equation

$$(1.9) \qquad \int_\Omega J\left(\frac{x-y}{g(y)}\right)\frac{u(y)}{g^d(y)}\,dy - b(x)u = 0. \quad \text{in} \quad \Omega.$$

Now let us state the precise assumptions that we impose on J, g and b. Throughout this paper and without further notice we will always assume that J b and g satisfy the following assumptions

$$J \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d), \tag{H1}$$

$$m_0\chi_{B(0,r_0)} \leq J \leq M_0\chi_{B(0,R_0)} \text{ for some positive constants } r_0, R_0, m_0, M_0, \tag{H2}$$

$$g \in L^\infty(\Omega), 0 \leq g \leq \beta, \frac{1}{g^d} \in L^p_{loc}(\Omega) \text{ with } p > 1, \tag{H3}$$

$$b \in C(\bar\Omega), b(x) > 0. \tag{H4}$$

Let us also denote by $\mathcal{S}$ the set of point where g is vanishing (i.e. $\mathcal{S} := \{x \in \bar\Omega | g(x) = 0\}$). Without loss of generality we will also assume that $R_0 = 1$. Note that when $\mathcal{S} \neq \emptyset$ the kernel $k(x,y) = J\left(\frac{x-y}{g(y)}\right)\frac{1}{g^d(y)}$ does not satisfy automatically the assumption of Cornea and Veselý (i.e. the condition (1.6)) moreover the operator $\mathcal{L}$ can not be rewritten as a jump operator like in the equation (1.7).

Under the above assumptions on J, g, and b, we first establish some kind of uniform estimate satisfied by the positive solutions of (1.9). More precisely, we prove the following

Theorem 1.3. *Let J, g, and b satisfying (H1-H4) and assume that $\Omega \cap \mathcal{S} \subset\subset \Omega$ and let $\omega \subset \bar{\Omega}$ be a compact set. Let us denote $\Omega(\omega)$ the following set*

$$\Omega(\omega) := \bigcup_{x \in \omega} B(x, \beta).$$

Then there exists a positive constant η^* such that for any $0 < \eta \leq \eta^*$ there exist a compact set $\omega' \subset\subset \Omega(\omega) \cap \Omega$ and a constant $C(J, \omega, \omega', b, g, \eta)$ such that the following assertions are verified

(i) $\{x \in \Omega(\omega) \cap W_\eta | d(x, \partial(\Omega(\omega) \cap W_\eta)) > \eta\} \subset \omega'$, *where* $W_\eta := \{x \in \Omega | g(x) > \eta\}$

(ii) *for all positive continuous solutions u of (1.9), the following inequality holds:*

$$u(x) \leq Cu(y) \quad \text{for all} \quad x \in \omega, y \in \omega' \cap \omega.$$

Under an additional assumption on the regularity of the compact set ω, we have a more precise estimate. Namely, we prove the following result.

Theorem 1.4. *Let J, g, and b satisfying (H1-H4) and assume that $\Omega \cap \mathcal{S} \subset\subset \Omega$. Let $\omega \subset \bar{\Omega}$ be a compact set which satisfies an uniform inner cone condition. Then there exists a positive constant η^* such that for any $0 < \eta \leq \eta^*$, there exists a constant $C(J, \omega, b, g, \eta)$ such that for all positive continuous solutions u of (1.9) the following inequality holds*

$$u(x) \leq Cu(y) \quad \text{for all} \quad x \in \omega, y \in \omega \cap \{y \in \Omega | g(y) > 2\eta\}.$$

Note that in the above Theorems (Theorems 1.3 and 1.4), no condition is required on the open set Ω. Therefore, the respective inequalities are holding as well when $\Omega = \mathbb{R}^n$. In this particular case the assumption on $\mathcal{S}$ can be weakened. More specifically instead of assuming that $\Omega \cap \mathcal{S} \subset\subset \Omega$ we can require that for any subset $\widetilde{\mathcal{S}}$ of $\mathcal{S}$, there exists a ball $B(x_0, R)$ such that

$$B(x_0, R) \cap \widetilde{\mathcal{S}} \subset\subset B(x_0, R).$$

Corollary 1.5. Let J, g, and b satisfying (H1-H4) and assume that $\mathcal{S}$ satisfies the above condition and let $\omega \subset \mathbb{R}^n$ be a compact set. Then there exists a positive constant η^* such that for any $0 < \eta \leq \eta^*$ there exists a constant $C(J, \omega, b, g, \eta)$ such that for all positive continuous solutions u of (1.9) the following inequality holds

$$u(x) \leq Cu(y) \quad \text{for all} \quad x \in \omega, y \in \omega \cap \{y \in \Omega | g(y) > 2\eta\}.$$

Observe that when the compact set $\omega \subset \Omega$ and the considered kernel $k(x, y)$ satisfies the condition (1.6) we recover the Harnack type estimate obtained by Cornea in [14, 15].

As consequence of Theorems 1.3 and 1.4 we get various uniform estimates on the positive solutions u of (1.9) when Ω is a bounded domain. More precisely for general bounded domains we have

Corollary 1.6. Assume J, g and b satisfy (H1-H4), let Ω be a bounded domain and assume that $\frac{1}{g^d} \in L^p(\Omega)$ for $p > 1$. Then there exists a positive constant η^* such that for any $0 < \eta \leq \eta^*$ there exists a compact set $\omega' \subset\subset \Omega$ and a constant $C(J, \Omega, \omega', b, g, \eta)$ such that the two following assertions hold

(i) $\{x \in \Omega | d(x, \partial W_\eta) > \eta\} \subset \omega'$

(ii) for all positive continuous solutions u of (1.9),

$$\sup_\Omega u \leq Cu(y) \quad \text{for all } y \in \omega'.$$

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Note that in this case, the assumption $\mathcal{S} \cap \bar{\Omega} \subset\subset \Omega$ is not any more required and the function g can vanish at the boundary.

In the case of a regular bounded domain and assuming that $g > \alpha$ the above inequality extends up to the boundary.

Corollary 1.7. Let J, g, and b satisfying (H1-H4) and assume that $g \geq \alpha > 0$ and Ω is a bounded domain which satisfies an uniform inner cone condition. Then there exists a constant $C(\Omega, g, J, b)$ such that for all positive continuous solutions u of (1.9) it holds

$$\sup_\Omega u \leq C \inf_\Omega u.$$

Note that, in this particular case, whereas the classical Harnack type inequality remains true for positive solutions of the uniformly elliptic equation (1.4) its extension up to the boundary (Corollary 1.7) does not. The validity of such an extension is a consequence of the nonlocal nature of the equation considered.

Our last result is an application of these estimates to the construction of a positive solution of (1.9) for a particular $b(x)$. Let us consider the equation

$$(1.10) \qquad \int_\Omega J\left(\frac{x-y}{g(y)}\right) \frac{u(y)}{g^n(y)} \, dy - a(x)u = 0 \quad \text{in} \quad \Omega,$$

where $a(x)$ is defined as follows

$$a(x) := \begin{cases} \int_\Omega J\left(\frac{y-x}{g(x)}\right) \frac{dy}{g^n(x)} & \text{if} \quad x \notin \mathcal{S} \\ 1 & \text{otherwise} \end{cases}$$

In the literature this equation corresponds to the nonlocal analogue of the usual eigenvalue problem for an elliptic operator with homogeneous Neumann boundary condition. For some particular situations, the existence of a positive solution is also well known. Indeed when $g \equiv cste$ then it is easy to see that for any J any constant is a solution of the problem (1.10). In this particular situation the kernel is of Markov type which is not true when $g \not\equiv cste$.

Another well known case is when Ω is a bounded domain and J is assumed to be a smooth (i.e. at least continuous) function. Then in such a case the equation (1.10) reduces to a homogeneous second kind Fredholm integral equation, well studied in the literature, see for example [37]. To our knowledge besides the two above situations the existence of a non trivial positive solution of (1.10) under some general assumptions on J, g and b has only been obtained in the case $\Omega = \mathbb{R}$ see [16].

Let us now state our result.

Theorem 1.8. *Let J, g, and b satisfying (H1-H4) and assume that $\Omega \cap \mathcal{S} \subset\subset \Omega$. Then there exists a positive continuous solution p of (1.10).*

2. COMMENTS AND STRAIGHTFORWARD GENERALISATIONS

Before going into the proofs of these results, let us make some comments and explain our strategy of proofs.

We first want to emphasize that the continuity assumption $H4$ made on b is not necessary and all the results can be proved assuming that $b \in L^\infty(\Omega)$ with $inf_\Omega b > 0$. We want also to point out that all the above results extend obviously to the kernels $k(x, y)$ of the form

$$k(x, y) = J\left(\frac{x_1 - y_1}{g_1(y)}; \frac{x_2 - y_2}{g_2(y)}; \ldots; \frac{x_n - y_n}{g_n(y)}\right) \frac{1}{\prod_{i=1}^n g_i(y)}.$$

We also remark that the assumption $J > c_0 \chi_{B(0,r_0)}$ cannot be removed easily and a generalisation of the Harnack type estimate for more general measure J seems delicate. Indeed it is well known that the Harnack type inequality is false for some discrete Laplacian Δ_h which corresponds to have in our framework $b = g \equiv 1$ and $J := \frac{1}{2}(\delta_h + \delta_{-h})$ where δ_h denotes the Dirac mass at the point h, see [35]. For example by taking $h = 2\pi$ we have $1 + cos(x)$ is a non negative solution of the average equation

$$u(x) = \int_{\mathbb{R}} J(x - y)u(y)\, dy = \frac{1}{2}(u(x + 2\pi) + u(x - 2\pi)).$$

Extension of these Harnack type inequality for positive measure J are currently under consideration.

We want also to emphasize even though the average equation (1.9) have some similarities with the average equations studied in the "Converse Restricted Mean Value Problem", the two equations are fundamentally different. Indeed whereas a function satisfying a Restricted Mean Value Property with respect to a measure μ verifies

$$(2.1) \qquad\qquad u(x) = \int_{B(x,\delta(x))} u(y)d\mu(y)$$

for some function $\delta(x)$, a solution u of (1.9) satisfies

$$u(x) = \int_{B(x,g(y))\cap\Omega} u(y)d\nu(y),$$

which in the case $g \not\equiv cste$ cannot be rewritten in the form (2.1) for some measure μ.

We also have recently observed that all our results on Harnack type inequality can be extended to the framework of super-median function (i.e u so that $\mathcal{L}[u] \leq 0$). In this framework, we have a Harnack type estimate of the form

Theorem 2.1. *Let J, g, and b satisfying (H1-H4). Assume that $\Omega \cap S \subset\subset \Omega$ and let $\omega \subset\subset \Omega$ be a compact set. Let $\Omega(\omega)$ denote the following set*

$$\Omega(\omega) := \bigcup_{x \in \omega} B(x, d_\omega),$$

where $d_\omega := d(\omega, \partial\Omega)$. Then there exists a positive constant η^* such that, for any $0 < \eta \leq \eta^*$, there exists a compact set $\omega' \subset\subset \Omega(\omega)$ and a constant $C(J, \omega, \omega', b, g, \eta)$ such that the following assertions are verified

    (i) *$\{x \in \Omega(\omega) \cap W_\eta | d(x, \partial(\Omega(\omega) \cap W_\eta)) > \eta\} \subset \omega'$, where $W_\eta := \{x \in \Omega | g(x) > \eta\}$*
    (ii) *for all positive continuous super median function u of (1.9) the following inequality holds:*

$$\frac{1}{b(x)} \int_\Omega J\left(\frac{x - z}{g(z)}\right) \frac{u(z)\, dz}{g^d(z)} \leq C\frac{1}{b(y)} \int_\Omega J\left(\frac{y - z}{g(z)}\right) \frac{u(z)\, dz}{g^d(z)} \quad \text{for all} \quad x \in \omega, y \in \omega' \cap \omega.$$

We have remarked that some of our results can be generalized easily to the situation where the function g is vanishing at the boundary of the set Ω provided that $g(x) < d(x, \partial\Omega)$. More precisely, we have

Theorem 2.2. *Let J, g, and b satisfying (H1-H4). Assume that $\mathcal{S} = \partial\Omega \cup \mathcal{S}_i$ with $\Omega \cap \mathcal{S}_i \subset\subset \Omega$ and $g(x) < d(x, \partial\Omega)$. Let $\omega \subset\subset \Omega$ be a compact set and let $\Omega(\omega)$ denote the following set*

$$\Omega(\omega) := \bigcup_{x \in \omega} B(x, d_\omega),$$

where $d_\omega := d(\omega, \partial\Omega)$. Then there exists a positive constant η^* such that, for any $0 < \eta \leq \eta^*$, there exist a compact set $\omega' \subset\subset \Omega(\omega)$ and a constant $C(J, \omega, \omega', b, g, \eta)$ such that the following assertions are verified

   (i) *$\{x \in \Omega(\omega) \cap W_\eta | d(x, \partial(\Omega(\omega) \cap W_\eta)) > \eta\} \subset \omega'$, where $W_\eta := \{x \in \Omega | g(x) > \eta\}$*
   (ii) *for all positive continuous solutions u of (1.9), the following inequality holds:*

$$u(x) \leq C u(y) \quad \text{for all} \quad x \in \omega, y \in \omega' \cap \omega.$$

Now let us explain our strategy to obtain such Harnack type inequality. It is mainly based on the following observation. For the harmonic functions (i.e functions u such that $\Delta u = 0$) it is well known (see [24, 27]) that they satisfy a mean value equality

$$u(x) = \int_{B(x,r)} u(y) \frac{dy}{|B(x,r)|},$$

which holds for any ball $B(x,r) \subset\subset \Omega$. An Harnack inequality is derived easily from this property. Our idea is then to view a positive solution u of (1.9) as a positive function satisfying some mean value equality

$$(2.2) \qquad\qquad u(x) = \frac{1}{b(x)} \int_\Omega u(y) \, d\mu(x, y),$$

for some given measure $d\mu$, and to use this formulation to obtain some uniform estimates depending only on ω, J and b. However, the later mean value property is fundamentally different from the one satisfied by harmonic functions in at least two ways. First, the measure $d\mu(x, y)$ is no more homogeneous and may be singular in the variable y. Second, the solution of equation (1.9) satisfies the mean value equality for the fixed domain Ω, whereas for harmonic functions the mean value equality holds for any ball compactly included in Ω. All the difficulty in obtaining such Harnack type estimates arises from these two differences. We note that our proofs differ from the one given in [14, 15] providing alternative proofs of these Harnack type estimates. In particular, from our analysis, we can extract an estimate of the constant C involved in the Harnack type inequality.

The paper is organized as follows. In the next section, we establish some general estimates that we use along our paper. Then we establish some uniform estimates satisfied by the positive solutions of (1.9). Next, in section 5, we prove the various Harnack type inequalities appearing in Theorems 1.3, and 1.4, and Corollaries 1.5 to 1.7. Finally, the last section is devoted to the construction of a positive eigenfunction (Theorem 1.8).

3. PRELIMINARIES

In this section, we prove some technical lemmas concerning some sets of positive functions satisfying a pointwise estimate. Similar estimates and inequalities can be found in [15]. Let us state our first lemma,

Lemma 3.1. *Let $\Omega \subset \mathbb{R}^n$ be a domain and let $x \in \Omega$ and $\eta > 0$ so that $B(x, 4\eta) \subset \Omega$. Let C_0 be a positive constant and define X the following set of functions*

$$X(x) := \left\{ u \in C(\Omega) \mid u \geq 0, \text{ such that } u(y) \geq C_0 \int_{B(y,\eta)} u(s)\, ds \text{ for all } y \in B(x, 2\eta) \right\}$$

Then for all $\frac{\eta}{4} \leq r \leq \eta$, there exists a positive constant $C_1(r, \eta, C_0) \leq 1$ so that for all $u \in X$ we have

$$\int_{B(x,r)} u(s)\, ds \geq C_1 \int_{B(x,r+\frac{\eta}{4})} u(s)\, ds.$$

Assume for the moment that Lemma 3.1 holds. Then we can derive the following uniform estimates

Lemma 3.2. *Let $\Omega \subset \mathbb{R}^n$, $\Sigma \subset\subset \Omega$ and $C_0 > 0$ be respectively a domain, a smooth connected compact set and a positive constant. Choose $\eta > 0$ such that*

$$\Omega_{2\eta} := \bigcup_{x \in \Sigma} B(x, 4\eta) \subset\subset \Omega$$

and consider $\widetilde{X}$ the following set of functions

$$\widetilde{X} := \left\{ u \in C(\Omega) \mid u \geq 0, \text{ such that } u(y) \geq C_0 \int_{B(y,\eta)} u(s)\, ds \text{ for all } y \in \Omega_\eta := \bigcup_{x \in \Sigma} B(x, 2\eta) \right\}$$

Then there exists two positive constants $C(\eta, C_0, \Sigma)$ and $N(\eta, \Sigma)$ so that

(i) for all $x, y \in \Sigma$ and for all $u \in \widetilde{X}$ we have

$$\int_{B(x,\frac{\eta}{4})} u(s)\, ds \geq C \int_{B(y,\frac{\eta}{4})} u(s)\, ds.$$

(ii) for all $x \in \Sigma$ and for all $u \in \widetilde{X}$ we have

$$\int_{B(x,\frac{\eta}{4})} u(s)\, ds \geq \frac{C}{N} \int_{\Sigma} u(s)\, ds.$$

Proof of Lemma 3.2:

Let us start with the proof of *(i)*. Observe that for any $z \in \Sigma$ we have by assumption

(3.1) $$B(z, 2\eta) \subset \Omega_\eta \text{ and } \widetilde{X} \subset X(z).$$

Therefore applying Lemma 3.1 respectively with $r = \frac{\eta}{4}$ and $\frac{\eta}{2}$ yields for all $u \in \widetilde{X}$ and for all $z \in \Sigma$

$$\int_{B(z,\frac{\eta}{4})} u(s)\, ds \geq C_1\left(\frac{\eta}{4}, \eta, C_0\right) \int_{B(z,\frac{\eta}{2})} u(s)\, ds,$$

$$\int_{B(z,\frac{\eta}{2})} u(s)\, ds \geq C_1\left(\frac{\eta}{2}, \eta, C_0\right) \int_{B(z,\frac{3\eta}{4})} u(s)\, ds.$$

Thus for all $u \in \widetilde{X}$ and for all $z \in \Sigma$ we have

(3.2) $$\int_{B(z,\frac{\eta}{4})} u(s)\, ds \geq C_2 \int_{B(z,\frac{3\eta}{4})} u(s)\, ds,$$

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with $C_2 := C_1\left(\frac{\eta}{4}, \eta, C_0\right) C_1\left(\frac{\eta}{2}, \eta, C_0\right)$.

Using now that Σ is a compact connected set then there exists a finite number of balls $B(z_i, \frac{\eta}{4})$ covering Σ. That is to say, for some $N(\Sigma, \eta) \in \mathbb{N}$,

$$\Sigma \subset \bigcup_{i=1}^{N} B(z_i, \frac{\eta}{4}).$$

Now let $x, y \in \Sigma$ be fixed . From the covering, we can find a finite sequence $(t_n)_{n \in \{0,\ldots,N_0\}}$ of elements of Σ so that

$$\begin{cases} N_0 \le N(\Sigma, \eta) + 2 \\ t_i \in \{x, y, z_1, \ldots, z_N\}, \; t_0 = x \quad \text{and} \quad t_{N_0} = y \\ \forall i \in \{0, \ldots, N_0 - 1\}, B(t_{i+1}, \frac{\eta}{4}) \subset B(t_i, \frac{3\eta}{4}) \end{cases}$$

Since for all $u \in \widetilde{X}$, $u \ge 0$, using (3.2) with $z = x$ and the definition of t_1, we deduce that for all $u \in \widetilde{X}$

$$\int_{B(x,\frac{\eta}{4})} u(s)\, ds \ge C_2 \int_{B(x,\frac{3\eta}{4})} u(s)\, ds \ge C_2 \int_{B(t_1,\frac{\eta}{4})} u(s)\, ds.$$

Then by induction, for all $u \in \widetilde{X}$ we achieve

$$\int_{B(x,\frac{\eta}{4})} u(s)\, ds \ge (C_2)^{N_0} \int_{B(y,\frac{\eta}{4})} u(s)\, ds.$$

Since $N(\eta, \Sigma)$ is independent of x and y and $N_0 \le N + 2$ we have for all $x, y \in \Sigma$ and for all $u \in \widetilde{X}$,

$$\int_{B(x,\frac{\eta}{4})} u(s)\, ds \ge (C_2)^{N+2} \int_{B(y,\frac{\eta}{4})} u(s)\, ds,$$

which proves *(i)*.

To obtain *(ii)*, we just have to remark that using the above covering of Σ and *(i)* for all $x \in \Sigma$ and all $u \in \widetilde{X}$ we have

$$\begin{aligned} \int_{B(x,\frac{\eta}{4})} u(s)\, ds &\ge \sum_{i=1}^{N} \frac{(C_2)^{N+2}}{N} \int_{B(z_i,\eta)} u(s)\, ds \\ &\ge \frac{(C_2)^{N+2}}{N} \int_{\bigcup_{i=1}^{N} B(y_i,\eta)} u(s)\, ds \\ &\ge \frac{(C_2)^{N+2}}{N} \int_{\Sigma} u(s)\, ds. \end{aligned}$$

$\square$

Let us now turn our attention to the proof of the technical Lemma 3.1.

Proof of Lemma 3.1

Since $r \leq \eta$, for all $z \in B(x, r)$ we have $B(z, \eta) \subset B(x, 2\eta)$ and by assumption for all $u \in X(x)$ we have

$$(3.3) \qquad \int_{B(x,r)} u(s)\, ds \geq C_0 \int_{B(x,r)} \left(\int_{B(z,\eta)} u(s)\, ds \right) dz.$$

Let us now consider the annulus $A := A(x, r', r)$, for some $0 < r' < r$ which will be chosen later on. Observe that for $\epsilon > 0$, A can be covered by a finite numbers of balls $B(z, r - r' + \epsilon)$, where $z \in \partial B(x, r')$. Namely, we have for some $N_{(r-r'+\epsilon)} \in \mathbb{N}$,

$$A \subset \bigcup_{i=1}^{N_{(r-r'+\epsilon)}} B(z_i, (r - r' + \epsilon)).$$

Observe that by construction, for any $N_{(r-r')+\epsilon} -$tuplet $(\widetilde{z}_1, \widetilde{z}_2, \ldots, \widetilde{z}_{N_{(n-r')+\epsilon}})$ such that $\widetilde{z}_i \in B(z_i, (r - r') + \epsilon) \cap A$ we have

$$A \subset \bigcup_{i=1}^{N_{(r-r'+\epsilon)}} B(\widetilde{z}_i, 2(r - r') + 2\epsilon).$$

Moreover, for $r - r' + \epsilon < \frac{\eta}{2}$ and for any $N_{(r-r')+\epsilon} -$tuplet $(\widetilde{z}_1, \widetilde{z}_2, \ldots, \widetilde{z}_{N_{(r-r')+\epsilon}})$ such that $\widetilde{z}_i \in B(z_i, (r - r') + \epsilon) \cap A$, we see that

$$(3.4) \qquad A(x, r', r' + \frac{\eta}{2} - [(r - r') + \epsilon]) \subset \bigcup_{i=1}^{N_{(r-r'+\epsilon)}} B(\widetilde{z}_i, \eta).$$

Indeed, let $(\widetilde{z}_1, \widetilde{z}_2, \ldots, \widetilde{z}_{N_{(n-r')+\epsilon}})$ be a $N_{(r-r')+\epsilon} -$tuplet such that $\widetilde{z}_i \in B(z_i, (r - r') + \epsilon) \cap A$ and take $y \in A(x, r', r' + \frac{\eta}{2} - [(r - r') + \epsilon])$. Then $d(y, \partial B(x, r')) \leq \frac{\eta}{2} - [(r - r') + \epsilon]$ and there exists $\bar{z} \in \partial B(x, r')$ so that

$$\|y - \bar{z}\| = d(y, \partial B(x, r')).$$

Since $\bar{z} \in \partial B(x, r')$, by construction there exists $z_j \in \partial B(x, r')$ so that

$$\bar{z} \in B(z_j, r - r' + \epsilon).$$

Choose $\widetilde{z}_j$ the corresponding element in the $N_{(r-r')+\epsilon} -$tuplet $(\widetilde{z}_1, \widetilde{z}_2, \ldots, \widetilde{z}_{N_{(n-r')+\epsilon}})$ and compute $\|y - \widetilde{z}_j\|$ then we have

$$\|y - \widetilde{z}_j\| \leq \|y - \bar{z}\| + \|\bar{z} - z_j\| + \|z_j - \widetilde{z}_j\|$$
$$\leq \frac{\eta}{2} + [(r - r') + \epsilon]$$
$$\leq \eta$$

since $[(r - r') + \epsilon] < \frac{\eta}{2}$. Thus $y \in B(\widetilde{z}_j, \eta)$ and (3.4) holds.

Let us fix $\epsilon = \frac{r}{8}$ and choose $r' = \frac{15r}{16}$. Let us also denote $\mu(S)$ the Lebesgue measure of a set S and consider $A_i := B(z_i, (r - r') + \epsilon) \cap A$. By construction, since $y_i \in \partial B(x, r')$, we have $\mu(A_i) = \mu(A_j)$ for all i, j and for each A_i from (3.3) we have for all $u \in X(x)$

$$(3.5) \qquad \int_{B(x,r)} u(s)\, ds \geq C_0 \int_{A_i} \left(\int_{B(z,\eta)} u(s)\, ds \right) dz.$$

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Therefore for all $u \in X(x)$, on each A_i there exists a point $\widetilde{z}_i \in A_i$ which can depend on u such that

$$(3.6) \qquad \int_{B(x,r)} u(s)\,ds \geq \frac{C_0}{\mu(A_i)} \int_{B(\widetilde{z}_i,\eta)} u(s)\,ds.$$

Using that $\mu(A_i) = \mu(A_j)$ for all i, j, we deduce that for each $u \in X(x)$ there exists a $N_{(r-r')+\epsilon}$−tuplet $(\widetilde{z}_1, \widetilde{z}_2, \dots, \widetilde{z}_{N_{(n-r')+\epsilon}})$ so that

$$\int_{B(x,r)} u(s)\,ds \geq \frac{C_0}{\mu(A_i)N_{(r-r'+\epsilon)}} \sum_{i=1}^{N_{(r-r'+\epsilon)}} \int_{B(\widetilde{z}_i,\eta)} u(s)\,ds$$

$$\geq \frac{C_0}{\mu(A_i)N_{(r-r'+\epsilon)}} \int_{\bigcup_{i=1}^{N_{(r-r'+\epsilon)}} B(\widetilde{z}_i,\eta)} u(s)\,ds.$$

Since by construction for each $u \in X(x)$ we have $\widetilde{z}_i \in B(z_i, (r-r')+\epsilon) \cap A$ for all i. Using the geometric condition (3.4), it follows that for all $u \in X(x)$

$$\int_{B(x,r)} u(s)\,ds \geq \frac{C_0}{\mu(A_i)N_{r-r'+\epsilon}} \int_{A(x,r',r'+\frac{\eta}{2}-[(r-r')+\epsilon])} u(s)\,ds.$$

Therefore,

$$\int_{B(x,r)} u(s)\,ds \geq \frac{C_0}{\mu(A_i)2N_{r-r'+\epsilon}} \int_{A(x,r',r'+\frac{\eta}{2}-[(r-r')+\epsilon])} u(s)\,ds + \frac{1}{2} \int_{B(x,r)} u(s)\,ds$$

$$\geq C_1 \int_{B(x,r)\cup A(x,r',r'+\frac{\eta}{2}-[(r-r')+\epsilon])} u(s)\,ds$$

$$\geq C_1 \int_{B(x,r'+\frac{\eta}{2}-[(r-r')+\epsilon])} u(s)\,ds,$$

$$\geq C_1 \int_{B(x,r+h)} u(s)\,ds,$$

where $C_1 := \min\{\frac{C_0}{2\mu(A_i)N_{r-r'+\epsilon}}, \frac{1}{2}\}$ and $h := \frac{\eta}{2} - [2(r-r') + \epsilon]$.

Since by construction $h = \frac{\eta}{2} - [2(r-r') + \epsilon] \geq \frac{\eta}{2} - \frac{r}{4} \geq \frac{\eta}{4}$ we achieve for all $u \in \widetilde{X}(x)$,

$$\int_{B(x,r)} u(s)\,ds \geq C_1 \int_{B(x,r+h)} u(s)\,ds \geq C_1 \int_{B(x,r+\frac{\eta}{4})} u(s)\,ds.$$

$\square$

Remark 3.3. Note that from our construction we can make the constant C_1 independent of r. Namely, by our choice of r' and ϵ $N_{r-r'+\epsilon} = N_1$ is invariant with r and we have

$$C_1 \geq \min\{\frac{C_0}{2\eta^d\mu(A(0,\frac{15}{16},1)\cap B(z,\frac{17}{16}))N_1}, \frac{1}{2}\},$$

where z is any point of $\partial B(0,\frac{15}{16})$ and d the dimension of space.

4. Local uniform estimates

In this section, we establish some local uniform estimates, which will play an essential role in deriving Harnack type inequalities.

Lemma 4.1. *Let $\Omega \subset \mathbb{R}^n$ and $u \in C(\Omega)$ be respectively a connected domain and a positive solution of (1.9). Let $\Omega' \subset \Omega$ be a compact set such that $g \geq \alpha > 0$ in Ω'. Then there exists $\epsilon^* > 0$ such that for all $\epsilon \leq \epsilon^*$, there exists Ω_ϵ and $C_\epsilon(\alpha, \beta, J, \epsilon, b)$ such that*

$$\int_{\Omega_\epsilon} u(y)\, dy \geq C_\epsilon \int_{\Omega'} u(y)\, dy.$$

Moreover, Ω_ϵ satisfies the following chain of inclusion

$$\left\{ x \in \Omega' | d(x, \partial\Omega') > \alpha\epsilon \right\} \subset \Omega_\epsilon \subset \left\{ x \in \Omega' | d(x, \partial\Omega') > \frac{\alpha\epsilon}{2} \right\}.$$

Proof :

Since u is a positive solution of (1.9), using that $\Omega' \subset \Omega$, we have

$$(4.1) \qquad \int_{\Omega'} J\left(\frac{x-y}{g(y)}\right) \frac{u(y)}{g^d(y)}\, dy - b(x)u(x) \leq 0 \qquad \text{in} \qquad \Omega.$$

The domain Ω' being compact and u continuous, we can integrate (4.1) over $\Omega_\epsilon \subset\subset \Omega'$ and we have

$$\int_{\Omega_\epsilon} \int_{\Omega'} J\left(\frac{x-y}{g(y)}\right) \frac{u(y)}{g^d(y)}\, dy \leq \int_{\Omega_\epsilon} b(x)u(x)\, dx.$$

Using that $g \geq \alpha > 0$ in Ω', Fubini's Theorem and setting $z = \frac{x-y}{g(y)}$, we end up with

$$(4.2) \qquad \int_{\Omega_\epsilon} b(x)u(x)\, dx \geq \int_{\Omega'} \frac{u(y)}{g^d(y)} \left(\int_{\Omega_\epsilon} J\left(\frac{x-y}{g(y)}\right) dx \right) dy$$

$$(4.3) \qquad \geq \int_{\Omega'} u(y) \left(\int_{\Omega_{\epsilon,y}} J(z)\, dz \right) dy$$

where $\Omega_{\epsilon,y} := \frac{\Omega_\epsilon - y}{g(y)}$. We claim that

***Claim** 4.1.* There exist Ω_ϵ and $c_0 > 0$ such that, for all $y \in \Omega'$,

$$\int_{\Omega_{\epsilon,y}} J(z)dz > c_0.$$

Observe that by proving the above claim the proof of the lemma is ended. Indeed, assuming it is true, then we derive from the above inequality

$$\int_{\Omega_\epsilon} b(x)u(x)\, dx \geq \int_{\Omega'} u(y) \left(\int_{\Omega_{\epsilon,y}} J(z)\, dz \right) dy,$$

$$\geq c_0 \int_{\Omega'} u(y)dy.$$

Hence,

$$\int_{\Omega_\epsilon} u(x)\, dx \geq \frac{c_0}{\|b\|_\infty} \int_{\Omega'} u(y)dy.$$

$\square$

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Proof of the claim

By assumption, since $J(0) > 0$ and J smooth, there exist $r_0 > 0$ and c_0 such that $\min_{B(0,r_0)} J > c_0$.

Fix $\epsilon \leq \min\{\frac{r_0}{2}; \frac{1}{4}\}$ such that the two sets

(4.4)
$$\Omega'_\epsilon := \{x \in \Omega' | \ d(x, \partial\Omega') \geq \epsilon\alpha\}$$

(4.5)
$$\widetilde{\Omega}'_\epsilon := \{x \in \bar{\Omega}' | \ d(x, \partial\Omega') \leq \frac{\epsilon\alpha}{2}\}$$

are non empty disjoint sets. Choose Ω_ϵ smooth (at least C^2) so that $\Omega'_\epsilon \subset\subset \Omega_\epsilon$ and $\Omega_\epsilon \cap \widetilde{\Omega}'_\epsilon = \emptyset$. By construction, we see that $\bar{\Omega}_\epsilon$ is compact and for all $y \in \Omega'$, $d(y, \Omega_\epsilon) < \epsilon\alpha$.

Again by construction we observe that for $\delta \leq \frac{\epsilon\alpha}{4}$, we also have

$$\forall \ z \in \Omega_\epsilon, \quad B(z, \delta) \subset \Omega'.$$

From the uniform regularity of Ω_ϵ, there exists a constant δ small enough, say $0 < \delta < \delta_1 \leq \frac{\epsilon\alpha}{4}$ where δ_1 only depends on the regularity of $\partial\Omega_\epsilon$, such that for all $z \in \Omega_\epsilon$, there exists $z' \in \Omega_\epsilon \cap B(z, \delta)$ satisfying

(4.6)
$$B(z', \frac{\delta}{8}) \subset B(z, \delta) \cap \Omega_\epsilon.$$

Now, pick $y \in \Omega'$. Since $\bar{\Omega}_\epsilon$ is compact, there exists $z_0 \in \bar{\Omega}_\epsilon$ such that $\|y - z_0\| = d(y, \Omega_\epsilon)$. Using (4.6), it follows that there exists z'_0 such that

(4.7)
$$B(z'_0, \frac{\delta}{8}) \subset B(z_0, \delta) \cap \Omega_\epsilon.$$

Recall that $\Omega_{\epsilon,y} = \frac{\Omega_\epsilon - y}{g(y)}$ thus from (4.7) it follows that

$$\frac{B(z'_0, \frac{\delta}{8}) - y}{g(y)} \subset \Omega_{\epsilon,y}.$$

Take now $s \in B(z'_0, \frac{\delta}{8})$ and let us compute $\frac{\|s - y\|}{g(y)}$:

$$\frac{\|s - y\|}{g(y)} \leq \frac{\|s - z'_0\| + \|z_0 - z'_0\| + \|z_0 - y\|}{\alpha}$$

$$\leq \frac{\delta}{8\alpha} + \frac{\delta}{\alpha} + \epsilon.$$

Since $\delta \leq \frac{\epsilon\alpha}{4}$ and $\epsilon \leq \frac{r_0}{2}$, we achieve

$$\frac{\|s - y\|}{g(y)} \leq (\frac{1}{64} + \frac{1}{8} + \frac{1}{2})r_0 \leq r_0.$$

Finally, let us compute $\int_{\Omega_{\epsilon,y}} J(z)dz$. From the above construction, we have

$$\int_{\Omega_{\epsilon,y}} J(z)dz \geq \int_{\frac{B(z'_0, \frac{\delta}{8}) - y}{g(y)}} J(z)\, dz$$

$$\geq c_0 \int_{\frac{B(z'_0, \frac{\delta}{8}) - y}{g(y)}} dz$$

$$\geq c_0\mu(B(0, \frac{\delta}{8\beta})).$$

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Since the above computation is independent of $y \in \Omega'$, the claim is proved.

$\square$

Remark 4.2. Observe that from the above computation, we have a certain degree of freedom over the parameter ϵ, which later can be chosen at our convenience.

Let us now show another important estimate.

Lemma 4.3. *Let $\Omega \subset \mathbb{R}^n$ be a connected set and u a positive continuous function satisfying* (1.9). *Let $\Omega' \subset \Omega$ be such that $g \geq \alpha > 0$ in Ω'. Then for any $\Omega'' \subset\subset \Omega'$ there exist δ and a constant $C(\beta, J, \alpha, \delta, b)$ such that*

$$\forall\, x \in \Omega_\delta := \bigcup_{x \in \Omega''} B(x, \delta), \quad u(x) \geq C \int_{B(x,\delta)} u(y)\, dy.$$

Proof:

Let $d := d(\Omega'', \partial\Omega')$. By assumption, one has $d > 0$. Since u is positive and $\Omega' \subset \Omega$, we deduce from (1.9) that at $x \in \Omega$ we have

$$b(x)u(x) \geq \int_{\Omega'} J\left(\frac{x - y}{g(y)}\right) \frac{u(y)}{g^d(y)}\, dy$$

$$\geq \int_{B(x,\beta) \cap \Omega'} J\left(\frac{x - y}{g(y)}\right) \frac{u(y)}{g^d(y)}\, dy.$$

Using that $J(0) > 0$, $\alpha \leq g \leq \beta$ in Ω' and $\|b\|_\infty < C$ we see that for δ small, say $\delta \leq \delta_1$, we have

$$u(x) \geq \int_{B(x,\delta) \cap \Omega'} J\left(\frac{x - y}{g(y)}\right) \frac{u(y)}{g^d(y)}\, dy$$

$$\geq \frac{\min_{B(0,\delta)} J}{\beta^n \|b\|_\infty} \int_{B_\delta(x) \cap \Omega'} u(y)\, dy.$$

Choosing $\delta < \min\{\frac{d}{2}, \delta_1\}$, it follows that for any $x \in \Omega_\delta$, $B(x, \delta) \subset \Omega'$. Hence, for all $x \in \Omega_\delta$, we have

$$u(x) \geq \frac{\min_{B(0,\delta)} J}{\beta^n \|b\|_\infty} \int_{B_\delta(x)} u(y)\, dy$$

and the Lemma is proved.

$\square$

5. HARNACK TYPE INEQUALITIES

We are now in position to prove the different Harnack type inequalities, Theorems 1.3 and 1.4 and Corollaries 1.7 and 1.5. The proof of the Corollaries come as a straightforward application of the main Theorems and thus left to the reader. Simple proofs of Theorem 1.3 can be obtained using Theorems 1.4, so let us first prove Theorem 1.4.

Proof Theorem 1.4:

Before we begin, let us make some remarks and introduce some notation. First observe that if the estimates in Theorem 1.4 hold for a given compact set $\omega \subset \Omega$, then they hold as well for any compact set $\widetilde{\omega} \subset \omega$. Indeed, since the estimates in Theorem 1.4 hold for ω, there

exists a positive constant $\eta^*(\omega)$ so that for any $0 < \eta \leq \eta^*(\omega)$ there exists a constant $C(\eta)$ such that for all positive solution u of (1.9) the following inequality holds:

$$u(x) \leq Cu(y) \quad \text{for all} \quad x \in \omega, y \in \omega \cap W_{2\eta},$$

where $W_\eta := \{y \in \Omega | g(y) \geq \eta\}$.

Using now that any positive solutions of (1.9) satisfies

$$\sup_{\widetilde{\omega}} u \leq \sup_{\omega} u \leq C(\eta) \inf_{W_{2\eta} \cap \omega} u \leq C(\eta) \inf_{W_{2\eta} \cap \widetilde{\omega}} u,$$

it follows that the estimates in Theorem 1.4 hold for $\widetilde{\omega}$.

From the above observation, we can restrict our attention to compact set $\omega \subset \Omega$ such that $\mathcal{S} \subset\subset \omega$. Fix now ω and let us define the following sets

$$\omega_\eta := \bigcup_{x \in \omega} B(x, \eta) \cap \Omega$$

$$Z_\eta := \{y \in \Omega | g(y) < \eta\}$$

$$W_\eta := \{y \in \Omega | g(y) \geq \eta\}.$$

Since $\frac{1}{g^d} \in L^p_{loc}$ with $p > 1$, we can choose η_1 small enough such that

(5.1)
$$\int_{\omega \cap Z_{\eta^*}} \frac{dy}{g^d(y)} \leq \frac{\inf_\omega b}{2\|J\|_\infty}.$$

Since $\mathcal{S} \subset\subset \omega$, we can choose η_1 smaller if necessary to achieve $\omega_{\eta_1} \cap Z_\eta \subset \omega$. Fix now, $0 < \eta \leq \eta_1$. We are now in a position to prove the Theorem. The proof follows essentially four steps.

Step 1: Now, define the following bounded set

$$\Omega(\omega) := \bigcup_{x \in \omega} B(x, \beta),$$

and set the measure $d\mu = \frac{dy}{g^d(y)}$, which is well defined since $\frac{1}{g^d} \in L^1_{loc}$. Since J is compactly supported, it follows that in ω, u satisfies

(5.2)
$$u(x) = \frac{1}{b(x)} \int_{\Omega(\omega) \cap \Omega} J\left(\frac{x - y}{g(y)}\right) u(y) d\mu(y)$$

(5.3)
$$= \frac{1}{b(x)} \int_{\Omega(\omega) \cap Z_\eta} J\left(\frac{x - y}{g(y)}\right) u(y) d\mu(y) + \frac{1}{b(x)} \int_{\Omega(\omega) \cap W_\eta} J\left(\frac{x - y}{g(y)}\right) u(y) d\mu(y).$$

Observe that for $x \in \omega$, $y \in Z_\eta \cap (\Omega(\omega) \setminus \omega_\eta)$, we have

$$\left|\frac{x - y}{g(y)}\right| \geq 1.$$

Therefore since $supp(J) \subset B(0, 1)$, it follows that for $x \in \omega$

$$\frac{1}{b(x)} \int_{\Omega(\omega) \cap Z_\eta} J\left(\frac{x - y}{g(y)}\right) u(y) d\mu(y) = \frac{1}{b(x)} \int_{\omega_\eta \cap Z_\eta} J\left(\frac{x - y}{g(y)}\right) u(y) d\mu(y)$$

and from (5.3) we get

(5.4)
$$u(x) \leq \frac{1}{b(x)} \int_{\omega_\eta \cap Z_\eta} J\left(\frac{x - y}{g(y)}\right) u(y) d\mu(y) + \frac{\|J\|_\infty}{\inf_\omega b(x)} \int_{\Omega(\omega) \cap W_\eta} u(y) d\mu(y).$$

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Since u is continuous and ω is compact, u achieves its maximum at some point, say $x_0 \in \omega$. At this point, from (5.4) we have:

$$(5.5) \qquad u(x_0) \le \frac{1}{b(x_0)} \int_{\omega_\eta \cap Z_\eta} J\left(\frac{x_0 - y}{g(y)}\right) u(y) d\mu(y) + \frac{\|J\|_\infty}{\inf_\omega b(x)} \int_{\Omega(\omega) \cap W_\eta} u(y) d\mu(y).$$

Using that $\omega_\eta \cap Z_\eta \subset \omega$ and (5.1), it follows that

$$(5.6) \qquad u(x_0) \le \frac{u(x_0)}{2} + \frac{\|J\|_\infty}{\inf_\omega b(x)} \int_{\Omega(\omega) \cap W_\eta} u(y) d\mu(y).$$

Therefore,

$$(5.7) \qquad u(x_0) \le \frac{2\|J\|_\infty}{\inf_\omega b(x)} \int_{\Omega(\omega) \cap W_\eta} u(y) d\mu(y).$$

Step 2: For any $\nu \in \mathbb{R}$, let us consider the set $\omega_\nu := \{x \in \omega | d(x, \partial\omega) \ge \nu\}$ and $C(x, \theta, a)$ be the cone issued from x with angle θ and height a. On one hand, since $\mathcal{S} \subset\subset \omega$, there exists $\nu_0 > 0$ such that $\omega \setminus \omega_{4\nu_0} \cap \mathcal{S} = \emptyset$. On the other hand, since ω satisfies an uniform inner cone condition, it follows that for ν small enough, say $\nu \le \nu^*$, there exists $r(\nu) > 0$ such that for any $x \in \omega \setminus \omega_\nu$, there exists $\bar{x} \in \omega$ such that

$$B(\bar{x}, r) \subset C_{x, \theta, a} \cap (\omega_\nu \setminus \omega_{4\nu})$$
$$B(\bar{x}, r) \subset B(x, \beta).$$

Let us now fix $\nu \le \min\{\nu_0, \nu^*\}$ and take $\eta^* := \min_{\Omega(\omega) \setminus \omega_{4\nu}} g$. By construction, $\eta^* > 0$.

Now take any $x \in \omega \setminus \omega_\nu$. From (1.9), using the uniform inner cone property, we have

$$(5.8) \qquad u(x) = \frac{1}{b(x)} \int_\Omega J\left(\frac{x-y}{g(y)}\right) \frac{u(y)}{g^d(y)} dy$$

$$(5.9) \qquad \ge \frac{1}{b(x)} \int_{C_{x,\theta,a} \cap B(x,\beta)} J\left(\frac{x-y}{g(y)}\right) \frac{u(y)}{g^d(y)} dy$$

$$(5.10) \qquad \ge \frac{1}{b(x)} \int_{B(\bar{x},r)} J\left(\frac{x-y}{g(y)}\right) \frac{u(y)}{g^d(y)} dy.$$

Recall that $g \ge \eta^*$ in $B(\bar{x}, r)$. Therefore, since $J(0) > 0$, $b > 0$, there exists δ_0 and C_0 independent of x such that $B(\bar{x}, \delta_0) \subset B(\bar{x}, r)$ and

$$(5.11) \qquad u(x) \ge C_0 \int_{B(\bar{x}, \delta_0)} u(y) \, dy.$$

Fix now $\eta \le \min\{\frac{\eta_1}{2}, \frac{\eta^*}{2}\}$ such that $W_\eta \setminus W_{2\eta} \subset\subset \omega_\nu$, and let

$$d := d\left(\omega_\nu \cap W_{2\eta}, \partial(\Omega(\omega) \cap W_\eta)\right).$$

By construction, we have $d > 0$. Indeed, since $\eta \le \eta^*$, we have $\partial(\Omega(\omega) \cap W_\eta) = \Gamma_1 \cup \Gamma_2$ where $\Gamma_1 \subset \overline{(\Omega(\omega) \setminus \omega_{\frac{\nu}{2}})}$ and $\Gamma_2 \subset \overline{(W_\eta \setminus W_{\frac{3\eta}{4}})}$. Therefore, for any $x \in \omega_\nu \cap W_{2\eta}$

$$d(x, \Gamma_1 \cup \Gamma_2) \ge d\left(x, \overline{(\Omega(\omega) \setminus \omega_{\frac{\nu}{2}})} \cup \overline{(W_\eta \setminus W_{\frac{3\eta}{4}})}\right) > 0.$$

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Step 3: Let ν and η be defined by the above steps. Since $d > 0$, choosing ϵ small enough, say $\epsilon \le \frac{d}{2\eta}$, it follows that

$$\omega_\nu \cap W_{2\eta} \subset \{x | d(x, \partial(\Omega(\omega) \cap W_\eta)) \ge \epsilon\eta\}.$$

Now, since $g \ge \eta$ in $\Omega(\omega) \cap W_\eta$, from Lemma 4.1 there exists ϵ^* so that for all $0 < \epsilon \le \epsilon^*$ there exists a non empty set Ω_ϵ and a constant $C_\epsilon(J, \eta, \Omega(\omega) \cap W_\eta, b)$ such that

$$\{x | d(x, \partial(\Omega(\omega) \cap W_\eta)) \ge \epsilon\eta\} \subset \Omega_\epsilon \subset \left\{x | d(x, \partial(\Omega(\omega) \cap W_\eta)) \ge \frac{\epsilon\eta}{2}\right\}$$

and

(5.12)
$$\int_{\Omega_\epsilon} u(y) dy \ge C_\epsilon \int_{\Omega(\omega) \cap W_\eta} u(y) dy.$$

Observe that by choosing $\epsilon \le \min\{\epsilon^*, \frac{d}{2\eta}\}$, we also have $\omega_\nu \cap W_{2\eta} \subset \Omega_\epsilon$.

We now fix $\epsilon \le \min\{\epsilon^*, \frac{d}{2\eta}\}$ and choose $\delta < \frac{\epsilon\eta}{8}$, and consider the set

$$\Omega_\delta := \bigcup_{x \in \Omega_\epsilon} B(x, 2\delta).$$

By construction, we have $\Omega_\epsilon \subset\subset \Omega_\delta \subset \Omega(\omega) \cap W_\eta$ and $g \ge \eta$ in Ω_δ. Therefore, from Lemma 4.3 there exists δ_1 and C_0' such that for any $x \in \Omega_\epsilon$, we have

(5.13)
$$u(x) \ge C_0' \int_{B(x, \delta_1)} u(s) ds.$$

So we have

(5.14)
$$u(x) \ge \min\{C_0, C_0'\} \int_{B(x, \delta_1)} u(s) ds.$$

Step 4: Take now $\delta^* \le \min\{\delta_0, \delta_1\}$, where δ_0 is defined in (5.11). Then by construction Ω_ϵ and u satisfies the assumption of Lemma 3.2 and by (ii) of Lemma 3.2 we end up with

(5.15)
$$u(x) \ge \min\{C_0, C_0'\} \int_{B(x, \delta^*)} u(s)\, ds \ge \frac{C(\min\{C_0, C_0'\})}{N} \int_{\Omega_\epsilon} u(s)\, ds.$$

Collecting the inequalities (5.7), (5.12) and (5.15), it follows that for all $x \in \Omega_\epsilon$ we have

(5.16)
$$u(x_0) \le \frac{2\|J\|_\infty}{\inf_\omega b(x) C_\epsilon} \int_{\Omega_\epsilon} u(s)\, ds$$

(5.17)
$$\le \frac{2\|J\|_\infty}{\inf_\omega b(x) C_\epsilon} \frac{N}{C(\min\{C_0, C_0'\})} \int_{B(x, \delta^*)} u(s)\, ds$$

(5.18)
$$\le \frac{2\|J\|_\infty}{\inf_\omega b(x) C_\epsilon} \frac{N}{C(\min\{C_0, C_0'\}) \min\{C_0, C_0'\}} u(x).$$

Observing that $\omega_\nu \cap W_{2\eta} \subset \Omega_\epsilon$, from equation (5.18) it follows that

(5.19)
$$u(x_0) \le \bar{C} u(x) \qquad \text{for all} \qquad x \in \omega_\nu \cap W_{2\eta}.$$

Now observe that from (5.4), we also get that for all $x \in \omega \setminus \omega_\nu$,

$$u(x) \ge C_0 \int_{B(\bar{x}, \delta_0)} u(s)\, ds,$$

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where $B(\bar{x}, \delta_0) \subset \omega_\nu \setminus \omega_{4\nu}$. Therefore, since $\bar{x} \in \omega_\nu \cap W_{2\eta} \subset \Omega_\epsilon$ and $\delta^* \leq \delta_0$, it follows that

(5.20)
$$u(x) \geq C_0 \int_{B(\bar{x}, \delta_0)} u(s)\, ds \geq \min\{C_0, C_0'\} \int_{B(\bar{x}, \frac{\delta^*}{2})} u(s)\, ds.$$

Using again Lemma 3.2 yields

(5.21)
$$u(x) \geq \frac{C(\min\{C_0, C_0'\})}{N} \int_{\Omega_\epsilon} u(s)\, ds.$$

As above, we can combine the inequalities (5.21), (5.12) and (5.3) to obtain

(5.22)
$$u(x_0) \leq \bar{C} u(x) \qquad \text{for all} \qquad x \in (\omega \setminus \omega_\nu).$$

Thus, from the inequalities (5.19) and (5.22) we get

(5.23)
$$u(x_0) \leq \bar{C} u(y) \qquad \text{for all} \qquad y \in \omega \cap W_{2\eta}.$$

Hence, we have

(5.24)
$$u(x) \leq \bar{C} u(y) \qquad \text{for all} \qquad x \in \omega,\ y \in \omega \cap W_{2\eta}.$$

$\square$

Let us now deal with the general estimate and prove the Theorem 1.3.

Proof of Theorem 1.3:

Let us first observe that the estimate is straightforward if the set $\omega \subset\subset \Omega$. Indeed, in such cases there exists always a regular compact set $\widetilde{\omega}$ such that $\omega \subset\subset \widetilde{\omega} \subset \Omega$ and the Theorem 1.4 applies. Therefore the only case left to analyse is when the domain ω touches the boundary of Ω. In such case and without any regularity assumption on the domain, we cannot derive the estimates from a simple argument. To obtain the estimate we use a similar argument as for the proof of Theorem 1.4. The proof here follows essentially three steps.

Step 1: First remark that as for the proof of Theorem 1.4, we can restrict our attention to a compact set $\omega \subset \Omega$ such that $\mathcal{S} \subset\subset \omega$. Now let us define the sets $\Omega(\omega) := \bigcup_{x \in \omega} B(x, \beta) \cap \Omega$ and W_η as in the above proof. Following a similar argument, for a point $x_0 \in \omega$ where u achieves its maximum and for small enough η, say $\eta \leq \eta^*$, we have

(5.25)
$$u(x_0) \leq \frac{2\|J\|_\infty}{\inf_\omega b(x)} \int_{\Omega(\omega) \cap W_\eta} u(y) d\mu(y).$$

From (1.9), using that u, J and g are non-negative, it follows that for all $x \in \Omega$ we have

(5.26)
$$u(x) \geq \frac{1}{b(x)} \int_{\Omega(\omega) \cap W_\eta} J\left(\frac{x-y}{g(y)}\right) u(y) d\mu(y).$$

Since $g \geq \eta$ in $\Omega(\omega) \cap W_\eta$, from Lemma 4.1 there exists $\epsilon^* > 0$ such that for all $0 < \epsilon \leq \epsilon^*$, there exists a non empty set $\Omega_\epsilon \subset\subset \Omega(\omega) \cap W_\eta$ and a constant $C_\epsilon(J, \eta, \Omega(\omega) \cap W_\eta, b)$ such that

$$\{x | d(x, \partial(\Omega(\omega) \cap W_\eta)) \geq \epsilon\eta\} \subset \Omega_\epsilon \subset \left\{x | d(x, \partial(\Omega(\omega) \cap W_\eta)) \geq \frac{\epsilon\eta}{2}\right\}$$

and

(5.27)
$$\int_{\Omega_\epsilon} u(y) d\mu(y) \geq C_\epsilon \int_{\Omega(\omega) \cap W_\eta} u(y) d\mu(y).$$

Recall that from the proof of Lemma 4.1, we also have $\epsilon \leq \epsilon^* \leq \frac{1}{4}$. Thus, we have

$$\{x | d(x, \partial(\Omega(\omega) \cap W_\eta)) \geq \eta\} \subset \{x | d(x, \partial(\Omega(\omega) \cap W_\eta)) \geq \epsilon\eta\} \subset \Omega_\epsilon.$$

Step 2: Choose now $\delta < \min\left\{\frac{\epsilon\eta}{8}, \frac{\epsilon}{8}\right\}$, where ϵ and η are defined by the previous Step, and consider the set

$$\Omega_\delta := \bigcup_{x\in\Omega_\epsilon} B(x, 2\delta).$$

By construction, $\Omega_\delta \subset \Omega(\omega) \cap W_\eta$ and $g \geq \eta$ in Ω_δ. Using Lemma 4.3, for any $x \in \Omega_\epsilon$ we have

$$(5.28) \qquad u(x) \geq C_0 \int_{B(x,\delta)} u(s)ds.$$

Step 3. As in the previous proof, by construction Ω_ϵ and u satisfies the assumption of Lemma 3.2 and we end up with

$$(5.29) \qquad u(x) \geq C_0 \int_{B(x,\delta)} u(s)\,ds \geq \frac{C(C_0)}{N} \int_{\Omega_\epsilon} u(s)\,ds.$$

Collecting (5.25), (5.26) and (5.29), it follows that for all $x \in \Omega_\epsilon$ we have

$$u(x_0) \leq \frac{2\|J\|_\infty}{\inf_\omega b(x)C_\epsilon} \frac{N}{C(C_0)C_0} u(x).$$

Therefore, for all $y \in \omega$ and $x \in \Omega_\epsilon$ we have

$$u(y) \leq \bar{C}u(x).$$

Hence, for all $y \in \omega$ and $x \in \Omega_\epsilon \cap \omega$ we have

$$u(y) \leq \bar{C}u(x).$$

$\square$

6. CONSTRUCTION OF NON TRIVIAL POSITIVE SOLUTION OF A PARTICULAR NONLOCAL EQUATION

In this section, we deal with the construction of a positive solution of (1.10) and prove Theorem 1.8.

Proof of Theorem 1.8:

We treat two cases

Case 1: Ω bounded.

First, let us assume that Ω is bounded and J is regular. Let us define the operator $T \in \mathcal{L}(C(\Omega))$ by

$$Tu := \frac{1}{a(x)} \int_\Omega J\left(\frac{x-y}{g(y)}\right) \frac{u(y)}{g^d(y)}\,dy,$$

where

$$a(x) := \begin{cases} \int_\Omega J\left(\frac{y-x}{g(x)}\right) \frac{dy}{g^d(x)} & \text{for} \quad x \notin \mathcal{S} \\ 1 & \text{otherwise.} \end{cases}$$

Since $\frac{1}{g^d(y)} \in L^1_{loc}$, T is a compact operator. Moreover T is positive since g, a and J are non-negative functions. Using now the Krein-Rutman Theorem, there exists an eigenvalue λ and a continuous eigenfunction $\phi > 0$ such that

$$\frac{1}{a(x)} \int_\Omega J\left(\frac{x-y}{g(y)}\right) \frac{\phi}{g(y)}\,dy = \lambda\phi.$$

Integrating the equation over Ω, it follows that $\lambda = 1$, and ϕ is our desired solution. To obtain a solution in the case $J \in L^\infty \cap L^1(\Omega)$, we proceed by regularisation. Let $(\rho_n)_{n \in \mathbb{N}}$ a sequence of C^∞ mollifier and consider the problem with the regular kernel $J_n := J \star \rho_n$. By the above argumentation, for each n there exists $\phi_n > 0$ solution of the regularized problem. Now using the global estimate of Corollary (1.7) we have

$$sup_\Omega \phi_n \leq C(J_n) \inf_\omega \phi_n.$$

Let us normalize ϕ_n so that $\inf_\omega \phi_n \geq 1$. From a carefully analysis of $C(J_n)$ on can show that this constant depends only on $C(J, \Omega)$. Thus $(\phi_n)_{n \in \Omega}$ is a uniformly bounded sequence of continuous function. Using now Arzela-Ascoli Theoreme and a standard diagonal extraction argument, there exists a convergent subsequence (ϕ_n) which converges locally uniformly to a positive continuous bounded function ϕ. Moreover, using Lebesgue monotone convergence Theorem, one can see that ϕ is our desired solution.

Case 2: Ω unbounded.

Assume now that Ω is any open set and let Ω_n be an increasing sequence of bounded subsets such that $\lim_{n \to \infty} \Omega_n = \Omega$. Since $\mathcal{S} \subset\subset \Omega$, we can also assume that for all $n \in \mathbb{N}$, $\mathcal{S} \cap \Omega_n \subset\subset \Omega_n$. Let ϕ_n denote the associated solution to Ω_n with the normalization $\phi(x_0) = 1$ for some fixed $x_0 \in \Omega_n$ that we will choose later on. Since $n \in \mathbb{N}$, $\mathcal{S} \subset\subset \Omega_n$ and $(\Omega_n)_{n \in \mathbb{N}}$ is an increasing sequence of sets, for some η_1 small we have

$$\bigcap_{n \in \mathbb{N}} (\Omega_n \cap W_{\eta_1}) = \Omega_0 \cap W_{\eta_1} \neq \emptyset.$$

Let us choose $x_0 \in \Omega_0 \cap W_{\eta_1}$.

Let us now fix $n \in \mathbb{N}$ and consider the sequence of functions $(\phi_{n+k})_{k \in \mathbb{N}}$. By construction, ϕ_{n+k} satisfies the equation

$$\int_{\Omega_{n+k}} J\left(\frac{x-y}{g(y)}\right) \frac{\phi_{n+k}(y)}{g^d(y)} \, dy - a_{n+k}(x)\phi_{n+k} = 0.$$

Since Ω_n is an increasing sequence of bounded sets, for any $k \in \mathbb{N}$ we have $\Omega_n \subset \Omega_{n+k}$. Using Theorem 1.3 with Ω_n and ϕ_{n+k}, it follows that for any $k \in \mathbb{N}$ there exists a constant η_k^*, such that for all $\eta \leq \eta_k^*$ there exists ω_{n+k}' and a constant $C_{n+k}(J, g, \eta, \|a_{n+k}\|_\infty, \beta, \Omega_n)$ such that

(6.1) $$\{x \in \Omega_{n+k} | d(x, \partial(\Omega_n \cap W_\eta)) > \eta\} \subset \omega_{n+k}'$$

(6.2) $$\sup_{\Omega_n} \phi_{n+k} \leq C_{n+k}(\eta)\phi_{n+k}(x) \quad \text{for all} \quad x \in \omega_{n+k}'.$$

For each $k \in \mathbb{N}$, let us choose η_k such that

$$\Omega_n \cap W_{\eta_1} \subset \{x \in \Omega_n | d(x, \partial W_{\eta_k}) > \eta_k\}.$$

Using the monotonicity of the sequence $(\Omega_n)_{n \in \mathbb{N}}$, it follows that

$$\Omega_0 \cap W_{\eta_1} \subset \Omega_n \cap W_{\eta_1} \subset \{x \in \Omega_n | d(x, \partial W_{\eta_k}) > \eta_k\} \subset \omega_{n+k}'.$$

Therefore, from the above set inclusion and (6.2), it follows that

(6.3) $$\sup_{\Omega_n} \phi_n \leq C_{n+k}(\eta_k)\phi_n(x_0) \leq C_{n+k}(\eta_k).$$

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Now, observe that the sequence of positive functions $(a_{n+k}(x))_{k\in\mathbb{N}}$ is increasing in Ω_n and uniformly bounded. The monotonicity property follows easily from the monotonicity of the Ω_n. Indeed, recall that for any $x \in \Omega_n \setminus \mathcal{S}$ we have

$$a_{n+k}(x) = \int_{\Omega_{n+k}} J\left(\frac{y-x}{g(x)}\right) \frac{dy}{g^d(x)}.$$

Therefore, using that $\Omega_n \subset \Omega_n + 1$ and that J, g are non negative functions, it follows that

$$a_{n+k}(x) = \int_{\Omega_{n+k}} J\left(\frac{y-x}{g(x)}\right) \frac{dy}{g^d(x)} \leq \int_{\Omega_{n+k+1}} J\left(\frac{y-x}{g(x)}\right) \frac{dy}{g^d(x)} = a_{n+k+1}(x).$$

On the other hand, for $x \in \mathcal{S}$, we have $a_n(x) = 1$ for all $n \in \mathbb{N}$. Thus, $a_{n+k} \leq a_{n+k+1}$ in Ω_n.

From the definition of a_n, we also get easily the uniform bound. For any $x \in \Omega_n \setminus \mathcal{S}$, using a change of variable we have

$$a_n(x) = \int_{\Omega_n} J\left(\frac{y-x}{g(x)}\right) \frac{dy}{g^d(x)} \leq \int_{\frac{\Omega_n - x}{g(x)}} J(z)\, dz \leq 1.$$

Using that $(a_n(x))_n$ is uniformly bounded independent of n and increasing in Ω_n, we can make the constant C_{n+k} independent of k. Therefore, for all $k \in \mathbb{N}$, ϕ_{n+k} is uniformly bounded in Ω_n. Now, since ϕ_{n+k} is uniformly continuous on Ω_n, using Arzela-Ascoli's Theorem we can extract from $(\phi_{n+k})_{k\in\mathbb{N}}$ a sub-sequence which converges uniformly in Ω_n. By a standard diagonal argument, we can extract from $(\phi_n)_{n\in\mathbb{N}}$ a sub-sequence which converges to a function ϕ uniformly on every compact subset ω of Ω. Using that J has compact support and the Lebesgue dominated convergence theorem, passing to the limit in the equation yields

$$\int_\Omega J\left(\frac{x-y}{g(y)}\right) \frac{\phi}{g(y)}\, dy - a(x)\phi = 0.$$

$\square$

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